Homework 1

MA 564

Due February 3, 2026

1. (a) Prove that there cannot exist an uncountable collection of pairwise disjoint open intervals in $\mathbb{R}$.

   (b) Recall that a real number is called *algebraic* if it is the root of a polynomial with integer coefficients. Prove that the set of algebraic real numbers $\mathbb{A}$ is countable.

   (c) Given two non-empty sets A and B, B is said to have greater cardinality than A if there exists an injection from A to B, and no injection from B to A. Construct a sequence of infinite sets $(A_n)_{n\in\mathbb{N}}$ so that A_{n+1} has greater cardinality than A_n for all $n \in \mathbb{N}$.

2. (a) Let $C(\mathbb{R})$ denote the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Define a map $I : C(\mathbb{R}) \to C(\mathbb{R})$ as follows:

   $$\forall f \in C(\mathbb{R}), \quad I(f)(x) = \int_0^x f(t)dt$$

   Is I one-to-one? onto? (No justification needed).

   (b) Let $f : X \to X$ be a function, and let $f^{\circ n} : X \to X$ be f composed with itself n times. Suppose for every $x \in X$, there exists $n \in \mathbb{N}$ (possibly dependent on x) such that $f^{\circ n}(x) = x$. Then prove that f is a bijection.

3. Prove the *Cantor-Schroeder-Bernstein Theorem*: Given sets A and B, if $f : A \to B$ and $g : B \to A$ are injective maps, then there exists a bijection from A to B.
   Hint: Let $\ell : B \to B$ be given by $\ell(x) = f(g(x))$. Call a point $b \in B$ a *descendent* of $b' \in B$ if $\ell^{\circ n}(b') = b$ for some non-negative integer n. Consider the function $h : A \to B$ defined as

   $$h(a) = \begin{cases} g^{-1}(a) & \text{if } f(a) \text{ is the descendent of a point } b' \notin f(A) \\ f(a) & \text{otherwise} \end{cases}$$

   Show that h is bijective.

4. Suppose (X, d) is a metric space.

   (a) Define $d'(x, y) = \min\{1, d(x, y)\}$ for every $x, y \in X$. Prove that d' is a metric on X and that $d \sim d'$.

(b) Define $d''(x,y) = \frac{d(x,y)}{1+d(x,y)}$ for every $x, y \in X$. Prove that d'' is a metric on X and that $d \sim d''$.

5. Let p be a prime number. Define $|.|_p : \mathbb{Q} \to \mathbb{Q}$ as follows:

$$|0|_p = 0.$$

For every $x \neq 0$ in $\mathbb{Q}$, x can be uniquely written as $\frac{p^k m}{n}$, where $k, m, n \in \mathbb{Z}$ and p does not divide m or n. Let $|x|_p = \frac{1}{p^k}$.

   (a) Show that $|x + y|_p \leq \max\{|x|_p, |y|_p\}$.

   (b) Prove that $|.|_p$ is a norm on $\mathbb{Q}$.

   (c) Let d_p be the metric induced by the norm $|.|_p$: i.e., $d_p(x, y) = |x - y|_p$ for all $x, y \in \mathbb{Q}$. Calculate $d_2(2^n, 0)$ for all $n \in \mathbb{N}$. What is $\lim_{n\to\infty} d_2(2^n, 0)$?

   (d) Give a specific example of x, y, z, p for which $d_p(x, z) < \max\{d_p(x, y), d_p(y, z)\}$.

6. Denote a point in $p \in \mathbb{R}^2$ as $p = (p_x, p_y)$. Consider $d_v : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$d_v(p,q) = \begin{cases} 1 & p_x \neq q_x \text{ or } |p_y - q_y| \geq 1 \\ |p_y - q_y| & p_x = q_x \text{ and } |p_y - q_y| < 1 \end{cases}$$

Prove that d_v is a metric on $\mathbb{R}^2$ and induces a topology that is finer than the standard topology.

7. For $n \in \mathbb{N}$, let $V^n = \{0,1\}^n = \{(a_1, \cdots, a_n) : a_i \in \{0,1\} \forall i\}$. That is, V^n is the set of all words of length n over $\{0, 1\}$. Consider $D_H : V^n \times V^n \to \mathbb{R}$ given by defining $D_H(a, b)$ to be the number of positions where a differs from b (i.e., the cardinality of $\{i : a_i \neq b_i\}$).

   (a) Show that D_H is a metric on V^n. This is called the *Hamming distance* between words.

   (b) Can you extend D_H to a metric on $\bigcup_{n\in\mathbb{N}} V^n$?

8. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

9. (a) Let $\{\mathscr{T}_\alpha\}$ be a family of topologies on a set X. Show that $\bigcap_\alpha \mathscr{T}_\alpha$ is a topology on X. Is $\bigcup_\alpha \mathscr{T}_\alpha$ a topology on X?

   (b) Let $\mathscr{A}$ be a basis for a topology on a set X. Show that the topology generated by $\mathscr{A}$ is the intersection of all topologies on X that contain $\mathscr{A}$.

10. Let X be an uncountable set, and let $\mathscr{T}$ be the cofinite topology on X (i.e., $A \in \mathscr{T} \iff X \setminus A$ is a finite set). Show that no point $x \in X$ has a countable basis of neighborhoods in $\mathscr{T}$ (this would then imply that the topological space $(X, \mathscr{T})$ is not first countable).