Homework 3

MA 564

Due March 10, 2026

1. Prove that the product of two Hausdorff spaces is Hausdorff, and that any subspace of a Hausdorff space is Hausdorff.

2. Show that a space X is Hausdorff if and only if the diagonal $\Delta = \{(x, x) : x \in X\}$ is a closed subset of $X \times X$.

3. Let X be a topological space. Show the following for any subset $A \subseteq X$.

   (a) A is open if and only if $\partial A = \mathrm{Cl}(A) \setminus A$

   (b) ∂A is empty if and only if A is both open and closed

4. Assume that each integer (except 1) is divisible by a prime but nothing else about prime numbers (this is equivalent to assuming that each natural number bigger than 1 can be factored into primes). For any integer a and natural number d, let

$$B_{a,d} = \{\cdots, a-2d, a-d, a, a+d, \cdots\} = \{a + kd : k \in \mathbb{Z}\}$$

   Additionally, let

$$\mathcal{B} = \{B_{a,d} : a \in \mathbb{Z} \text{ and } d \in \mathbb{N}\}$$

   (a) Prove that $\mathcal{B}$ is a basis for a topology on $\mathcal{T}$ on $\mathbb{Z}$.

   (b) Show that any non-empty open set in this topology is infinite.

   (c) Show that each set $B_{a,d}$ is both open and closed in $(\mathbb{Z}, \mathcal{T})$.

   (d) What is the set $\bigcup\{B_{0,p} : p \text{ is a prime number}\}$? Prove that this set is closed in $(\mathbb{Z}, \mathcal{T})$.

   (e) What does the previous part tell you about the cardinality of the set of prime numbers? (Remark: this proof is due to Fürstenberg)

5. For each statement below, provide either a proof or a counter-example.

   (a) If D is dense in a topological space $(X, \mathcal{T})$ and $\mathcal{T}^*$ is another topology on X that is finer than $\mathcal{T}$, then D is dense in $(X, \mathcal{T}^*)$.

   (b) If $A \subseteq X$ and D is dense in X, then $A \cap D$ is dense in the subspace topology of A.

6. Let X and Y be topological spaces and $f : X \to Y$ be a map. Define the *graph of* f to be the subset

$$\Gamma(f) = \{(x, f(x)) : x \in X\} \subseteq X \times Y$$

Prove that f is continuous if and only if the map $h : X \to \Gamma(f)$ given by $h(x) = (x, f(x))$ is a homeomorphism.

7. Let X be a topological space. A family of sets $\mathcal{F} = \{B_\alpha \subseteq X : \alpha \in \Lambda\}$ is said to be *locally finite* if each $x \in X$ has a neighborhood N such that $N \cap B_\alpha \neq \emptyset$ for only finitely many α's.

   (a) Prove that if $\mathcal{F}$ is locally finite, then $\mathrm{Cl}\left(\bigcup_{\alpha\in\Lambda} B_\alpha\right) = \bigcup_{\alpha\in\Lambda} \mathrm{Cl}(B_\alpha)$.

   (b) With the same assumption on $\mathcal{F}$, show that if each B_α is closed, then the union of all the B_α's is closed.

   (c) (*The Pasting Lemmas*) Assume that $\mathcal{F}$ is a family of sets whose union is X. Let Y be a topological space and suppose $f_\alpha : B_\alpha \to Y$ is a continuous function for each $\alpha \in \Lambda$ and that for each non-empty intersection $B = B_\alpha \cap B_\beta$, the functions f_α and f_β agree on B (i.e., $f_\alpha(x) = f_\beta(x)$ for all $x \in B$). Consider the unique function $f : X \to Y$ so that $f(x) = f_\alpha(x)$ for all $x \in B_\alpha$.

      i. Show that if all the B_α's are open, then f is continuous.

      ii. Show that if there are only finitely many B_α's and they are all closed, then f is continuous.

      iii. Show that if $\mathcal{F}$ is a locally finite family of closed sets, then f is continuous.

8. Let $x_n \to x$ and $y_n \to y$ in $\mathbb{R}$ with the standard topology. Show that

   - $x_n + y_n \to x + y$
   - $x_n - y_n \to x - y$
   - $x_n y_n \to xy$
   - provided each $y_n \neq 0$ and $y \neq 0$, we have $\frac{x_n}{y_n} \to \frac{x}{y}$.

9. Let X be a topological space and Y be a metric space.

   (a) Suppose $f_n : X \to Y$ is a sequence of continuous functions that converge uniformly to a continuous function $f : X \to Y$. Suppose $x_n \to x$ in X. Then prove that $f_n(x_n) \to f(x)$.

   (b) Give an example where, if we replace uniform convergence of f_n to f with pointwise convergence, then the above property does not hold for some $x_n \to x$ in X (Hint: Try functions from $[0, 1]$ to $\mathbb{R}$).

10. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at exactly one point.