Homework 4

MA 564

Due March 31, 2026

1. Let $X = \{a, b, c\}$. Note that $T = \{\emptyset, \{a, b\}, \{c\}, X\}$ defines a topology on X. Show that X is normal, but not Hausdorff.

2. (a) Let $p : X \to Y$ be a continuous map. Show that if there exists a continuous map $f : Y \to X$ such that $p \circ f : Y \to Y$ is the identity map on Y, then p is a quotient map.

   (b) If $A \subseteq X$, a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Show that a retraction is a quotient map.

3. (a) Let $p : X \to Y$ be an open map (i.e., for every $U \subseteq X$ open, the set $p(U)$ is open). Show that if A is open in X, then the map $q : A \to p(A)$ obtained by restricting p is an open map.

   (b) Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be projection to the first coordinate. Let $A \subseteq \mathbb{R} \times \mathbb{R}$ be the set of all points (x, y) for which either $x \geq 0$ or $y = 0$ or both. Let $q : A \to \mathbb{R}$ be obtained by restricting the domain of π_1. Show that q is a quotient map. Is q an open map?

4. Suppose A and B are both closed subsets of a topological space X. Prove that $A \setminus B$ is separated from $B \setminus A$. Do the same assuming instead that A and B are open.

5. Suppose $X = A \cup B$ where $A \setminus B$ and $B \setminus A$ are separated.

   (a) Prove that for any $C \subseteq X$, $\mathrm{Cl}_X C = \mathrm{Cl}_A(A \cap C) \cup \mathrm{Cl}_B(B \cap C)$.

   (b) Conclude that C is closed if $C \cap A$ is closed in A and $C \cap B$ is closed in B.

   (c) Conclude that C is open if $C \cap A$ is open in A and $C \cap B$ is open in B.

   (d) Prove that if $f : X \to Y$ is a function (here Y is some topological space) such that $f|_A$ and $f|_B$ are continuous, then f is continuous.

6. Let (X, d) and (Y, d') be two connected metric spaces. Suppose $k > 0$ and that $(a, b) \in X \times Y$. Let $K = \{(x, y) \in X \times Y : d(x, a) \leq k \text{ and } d'(y, b) \leq k\}$.

   (a) Give an example to show that the complement of K in $X \times Y$ might not be connected.

(b) Prove that the complement of K in $X \times Y$ is connected if (X, d) and (Y, d') are unbounded spaces (a metric space (X, d) is unbounded if for every $M > 0$ there exist $x, y \in X$ such that $d(x, y) > M$).

7. (a) Show that in the cofinite topology on $\mathbb{R}$, every subspace is compact.

   (b) In the cocountable topology on $\mathbb{R}$, is $[0, 1]$ compact?

8. Let A and B be disjoint compact subsets of a Hausdorff space X. Prove that there exist disjoint open sets $U \supseteq A$ and $V \supseteq B$.

9. (a) Show that $\mathbb{R}_K$ is connected.

   (b) Show that $[0, 1]$ is not compact as a subspace of $\mathbb{R}_K$.

10. A topological space X is said to be limit point compact if for every infinite subset $A \subseteq X$, we have $L(A) \neq \emptyset$. Prove that if X is compact, then X is limit point compact. Give an example to show that the converse does not hold.