Homework 5

MA 564

Due April 14, 2026

1. A space X is said to be

   - *locally connected* if every $x \in X$ has a basis of neighborhoods $\{U_\alpha\}_{\alpha \in \Lambda}$ in which each U_α is connected.
   - *locally path connected* if every $x \in X$ has a basis of neighborhoods $\{U_\alpha\}_{\alpha \in \Lambda}$ in which each U_α is path connected.

   Prove that if X is connected and locally path connected, then X is path connected.

2. Find a connected but not path connected subset of the standard square in $\mathbb{R}^2$ which contains a pair of points from parallel sides.

3. Prove or disprove: a nonempty product $X \times Y$ is totally disconnected iff both X and Y totally disconnected.

4. A space X is said to be locally compact at a point $x \in X$ if there is a compact subspace C of x that contains some neighborhood of x. We say X is locally compact if it is locally compact at each of its points.

   (a) Prove that $\mathbb{R}$ is locally compact.

   (b) Prove that $\mathbb{Q}$ is not locally compact.

   (c) Prove that if Z is a compact Hausdorff space and $X \subseteq Z$ is such that $Z \setminus X$ is singleton, then Z is locally compact.

5. Recall that a topological space is said to be

   - *regular*, if for every closed set A and every point x not in A, there exist disjoint open sets U containing A and V containing x.
   - *normal*, if for every pair of disjoint closed sets A and B, there exist disjoint open sets U containing A and V containing B.

   (a) Prove that every metrizable space is normal.

   (b) Prove that every compact Hausdorff space is normal.

   (c) Prove that every regular second countable space is normal.

6. Prove that the product of countably many metric spaces is metrizable.

7. In this exercise our goal is to prove the following:

   Theorem (Urysohn's metrization theorem)**.** *Every second countable Hausdorff regular space is metrizable.*

   Let X be our space. Here is the strategy:

   (a) Using problem 5(c) and the Urysohn Lemma (stated in class), show that there exists a sequence of continuous functions $f_n : X \to [0,1]$ such that for every $x \in X$ and neighborhood U of x, there exists $n \in \mathbb{N}$ (depending on x) such that $f_n(x) > 0$ and $f_n\big|_{(X\setminus U)} \equiv 0$.

   (b) Let $F : X \to \mathbb{R}^{\mathbb{N}}$ be the function $F(x) = (f_1(x), f_2(x), \cdots)$ where $f_1, f_2, \cdots$ are as above. Prove that when $\mathbb{R}^{\mathbb{N}}$ is equipped with the product topology, this function is a topological imbedding.

   (c) Note that problem 6 implies that $\mathbb{R}^{\mathbb{N}}$ with the product topology is metrizable. Show that X is metrizable.

8. Let X be a locally compact Hausdorff space. Construct a new space Y as follows: let ∞ denote a point that is not in X. Then $Y = X \cup \{\infty\}$, and consider the following collection $\mathcal{T}$ of subsets of Y:

$$\mathcal{T} = \{U \subseteq X : U \text{ is open in } X\} \cup \{Y \setminus C : C \subseteq X \text{ is compact}\}$$

   (a) Prove that $\mathcal{T}$ is a Hausdorff topology on Y.

   (b) Prove that $(Y, \mathcal{T})$ is compact. The space Y is called *the one-point compactification* of X.

   (c) Show that if Y' is any other compact Hausdorff space with the property that $X \subseteq Y'$, and $Y' \setminus X$ is singleton, then Y' is homeomorphic to Y.

9. Prove that the one-point compactification of $\mathbb{R}$ is homeomorphic to $\mathbb{S}^1$.

10. Show that if X_1 and X_2 are locally compact Hausdorff spaces and $f : X_1 \to X_2$ is a homeomorphism, then f extends to a homeomorphism between their one-point compactifications.