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SEQUENCES

Def: Let X be a set. A "sequence" in X is a function $x: \mathbb{N} \longrightarrow X$ i.e., an infinite collection of points $x_1, x_2, x_3, \ldots$ ($x_n = x(n)$).

Def: Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ "converges to"/"tends to" x and write $x_n \longrightarrow x$, if for every neighborhood U of x, $\exists N \in \mathbb{N}$ s.t. $x_n \in U$ $\forall n \geq N$.

Eg: 1. In $\mathbb{R}$ with the std. topology,

$$1/n \longrightarrow 0,$$

$$\frac{1}{n^{1+\varepsilon}} \longrightarrow 0 \quad \forall \varepsilon > 0$$

$$\left(1+\frac{1}{n}\right)^n \longrightarrow e$$

2. Sequences in general spaces may not converge at all, or may converge to more than one point.

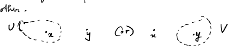
- ★ In $\mathbb{R}$ with the std topology, the sequence $X_n = n$ does not converge.
- ★ In the indiscrete topology $\mathcal{T} = \{\phi, X\}$ on any set X, every sequence converges to every point.
- ★ Line with doubled origin: Consider the non Hausdorff space $\mathbb{R}\setminus\{0\} \cup \{0_+, 0_-\}$ introduced in a previous lecture. Here, $1/n \longrightarrow 0_+, 0_-$.

Separation Axioms

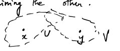
These are properties that measure how "separated" the closed sets of a topological space are.

A topological space $(X, \mathcal{T})$ is said to be:

★ T_0, if $\forall x, y \in X$, $x \neq y$, $\exists$ an open nbhood of at least one of the points that does not contain the other.



★ T_1, if $\forall x, y \in X$, $x \neq y$, both points contain nbhoods not containing the other.



★ T_2, if $\forall x, y \in X$, $x \neq y$, both points have nbhoods that are disjoint.

Eg:

1. A topological space that is T_0, but not T_1:
   $\mathbb{R}$, with a topology where the open sets are of the form (a, ∞).
   Given $x \neq y$ (WLOG $x < y$).
   $\left(\frac{x+y}{2}, \infty\right)$ contains y but not x.
   However it is not possible to find an open set containing x but not y.

2. Space that is T_1, but not T_2
   Cofinite topology on $\mathbb{R}$:
   No two open sets can be disjoint so this space is not T_2.
   Given $x \neq y$ in $\mathbb{R}$, $\mathbb{R}\setminus\{x\}$ is an open nbhood of y that does not contain x, and $\mathbb{R}\setminus\{y\}$ is an open nbhood of x not containing y.

Remark: $T_2 \Rightarrow T_0 \Rightarrow T_1$. [There are T_n spaces for $n \geq 3$, but we will discuss these later]

Sequences and Limit points

Fix a T_1-space $(X, \mathcal{T})$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, and suppose $x \in X$.

Prop: For any $A \subseteq X$, $x \in L(A) \Leftrightarrow$ every nbhood of x contains infinitely many points of A.

Proof:

Prop: Let $A = \{x_1, x_2, x_3, \ldots\}$.
Then $x \in L(A) \Leftrightarrow \exists$ a subsequence $x_{n_k} \longrightarrow x$.

Proof:

Prop: If X is T_2 (Hausdorff), any sequence of points converges to at most one point in X.

Proof: