Homework 1

MA 771

Due February 6 2025

Please scan or type your homework and hand/email it to me by 5pm on the due date.

1. Given a real number $\lambda > 0$, find the largest closed interval $I = [a, b]$ centered at a given point x_0 on the real line such that $f : I \longrightarrow \mathbb{R}$ given by $f(x) = x^2$ is a $\lambda-$contraction.

2. Show that the following statements are equivalent:
   - the endomorphism of the torus $T_A : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ is invertible
   - $x \in \mathbb{Z}^n$ if and only if $Ax \in \mathbb{Z}^n$, for every $x \in \mathbb{Z}^n$
   - $|\det A| = 1$.

3. Let $m > 1$ be an integer and let $E_m : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the expanding map $x \mapsto mx \pmod 1$.
   - Prove that the set of all periodic points of E_m is dense in $\mathbb{S}^1$.
   - A point $x \in \mathbb{S}^1$ is pre-periodic under E_m if $y = E_m^{\circ k}(x)$ is a periodic point for some $k \geq 0$. Show that the set of pre-periodic points under E_m is $\mathbb{Q}/\mathbb{Z}$.

4. Suppose that (X, d) is a compact metric space and $f : X \longrightarrow X$ is such that

$$d(f(x), f(y)) < d(x, y)$$

   for any $x \neq y$. Prove that f has a unique fixed point $x_0 \in I$ and $\lim_{n\to\infty} f^{\circ n}(x) = x_0$ for any $x \in X$.

5. Prove that the square Sierpinski carpet $\mathcal{J}$ is the set of points $(x, y) \in [0, 1] \times [0, 1]$ for which there exist ternary expansions $.x_1x_2x_3\ldots.$ for x and $.y_1y_2y_3\ldots$ for y such that for every $n \in \mathbb{N}$, either $x_n \neq 1$ or $y_n \neq 1$.