Homework 2

MA 771

Due February 20 2025

Please scan or type your homework and hand/email it to me by 5pm on the due date.

1. Let Σ_A be the space of sequences over a finite alphabet $A \subset X$, and let

$$d(s,t) = \sum_{j=1}^{\infty} \frac{d_X(s_j, t_j)}{|A|^{j-1}}$$

be the usual metric on A. The topology on Σ_A induced by d is called the *standard cylinder topology.*

- For $s \in \Sigma_A$ and $N \in \mathbb{N}$, define the *cylinder set* $U(s,N) = \{t \in \Sigma_A : s_j = t_j \text{ for } j = 1, 2, \cdots, N\}$. Prove that the collection of cylinder sets $\{U(s,N) : s \in \Sigma_A, N \in \mathbb{N}\}$ forms a basis for the above topology on Σ_A.
- Prove that the left shift operator $\sigma : \Sigma_A \longrightarrow \Sigma_A$ is topologically mixing.

2. Let $f(x) = 2x(1-x)$ and $g(x) = 5x(1-x)$. Show that the dynamical systems $(\mathbb{R}, f)$ and $(\mathbb{R}, g)$ are not topologically conjugate.

3. Prove that if $f, g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ are expanding maps of degree 2, then the dynamical systems $(\mathbb{S}^1, f)$ and $(\mathbb{S}^1, g)$ are topologically conjugate.

4. Let $A = (a_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with real entries. Suppose the linear map $x \mapsto Ax$ is a contraction on $\mathbb{R}^n$ (i.e., $||A|| < 1$), prove that

$$\mathbf{tr}\ A = \sum_{j=1}^{n} a_{jj} < n.$$

5. Let (X,d) be a metric space and suppose $f : X \longrightarrow X$ is an isometry (i.e., $d(f(x), f(y)) = d(x,y)$ for all $x, y \in X$). Show that f is not topologically mixing.

6. Let $(G, \cdot)$ be a metrizable compact topological group. Let $g_0 \in G$ be an element for which the translation $L_{g_0} : h \mapsto g_0 \cdot h$ is topologically transitive.

- Prove that the grand orbit $(L_{g_0}^{\circ n}(h))_{n \in \mathbb{Z}}$ is dense in G for every $h \in G$.
- Use the above to show that the cyclic subgroup of G generated by g_0 is dense in G.

- Prove that G is Abelian.
  Hint: The multiplication map on G is continuous. Using everything you did above, prove that for any $g, h \in G$, the elements $g \cdot h$ and $h \cdot g$ are arbitrarily close to each other in the metric of G.