Homework 4

MA 771

Due March 27 2025

Please scan or type your homework and hand/email it to me by 5pm on the due date.

1. Let $\Omega = \{\overline{s} = \cdots s_{-2}s_{-1}.s_0s_1s_2\cdots : s_i \in \{0,1\} \forall i \in \mathbb{Z}\}$ be the space of bi-infinite sequences over $\{0,1\}$. Consider the map $M : \Omega \to \Omega$ defined by

$$M(\cdots s_{-2}s_{-1}.s_0s_1s_2\cdots) = \cdots s_2s_1s_0.s_{-1}s_{-2}\cdots$$

   (a) Prove that M is an *involution*, i.e., $M \circ M = \text{id}$.

   (b) Additionally, let σ be the left shift map: $\sigma(\cdots s_{-2}s_{-1}.s_0s_1s_2\cdots) = \cdots s_{-1}s_0.s_1s_2\cdots$. Prove that $\sigma \circ M \circ \sigma = M$, and use this to show that $\sigma = U \circ M$ where $U : \Omega \to \Omega$ is a map which satisfies $U \circ U = \text{id}$.

   (c) Let $s = \cdots s_{-1}.s_0s_1\cdots$ be a sequence which is fixed by M. Suppose that $\sigma^{\circ n}(s)$ is also fixed by M. Prove that $s \in \text{Per}_{2n}(\sigma)$.

2. A function $F : \mathbb{R}^n \to \mathbb{R}$ is called an *integral* for a linear map L if $F \circ L(z) = F(z)$ for all $z = (x, y) \in \mathbb{R}^n$, i.e., F is constant along orbits of L.

   (a) Show that $F(z) = x^2 + y^2$ is an integral for

$$L(z) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} z$$

   (b) Construct a (non-trivial) integral for the linear map

$$L(z) = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} z$$

3. Consider the *tent map* $T : [0,1] \to [0,1]$ defined as

$$T(x) = \begin{cases} 2x & x \leq \frac{1}{2} \\ 2(1-x) & x > \frac{1}{2} \end{cases}$$

   (a) Prove that for any interval $I \subset [0,1]$, we have $T^{\circ N}(I) = [0,1]$ for some $N \geq 0$.

   (b) Show that T is topologically conjugate to $f(x) = 4x(1-x)$ on $[0,1]$.

4. Describe the dynamics of the linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ below. Identify precisely the stable and unstable sets.

$$L(x) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} x$$

5. Find a horseshoe in Arnold's cat map.