Homework 5

MA 771

Due April 28 2025

Please scan or type your homework and hand/email it to me by 5pm on the due date.

1. Let m, n be natural numbers such that $2 \leq m < n$. Let $A = \{0, 1, \cdots, m-1\}$ and $B = \{0, 1, \cdots, n-1\}$. Show that there exists a surjective continuous map $h : \Sigma_B \to \Sigma_A$ such that $h \circ \sigma = \sigma \circ h$. Is h invertible?

2. Suppose $X = \bigcup_i X_i$ is compact, $f : X \to X$ such that each X_i is closed in X and satisfies $f(X_i) = X_i$. Show that $h_{\text{top}}(f) = \sup_i h_{\text{top}}(f\big|_{X_i})$.

3. (a) Suppose $T : X \to X$ is a continuous transformation of a topological space X,and µ is a finite $T-$invariant Borel measure on X with $\text{supp}\mu = X$. Show that every point is non-wandering.

   (b) Show that an isometry of a compact metric space is not mixing for any invariant Borel measure whose support is not a single point. In particular, circle rotations are not mixing.

4. Let A be an $n \times n$ matrix over $\mathbb{R}$ and assume that A has only real eigenvalues. In this exercise we will prove that for every $\delta > 0$ there is a norm $|| \cdot ||$ on $\mathbb{R}^n$ such that $||A|| < r(A) + \delta$ by the following sequence of steps.

   - Let λ be an eigenvalue of A with multipicity k. Let $\Delta = A - \lambda \,\text{Id}$. Define the generalized eigenspace
   $$E_\lambda = \{v \in \mathbb{R}^n : \Delta^k v = 0\}$$
   Prove that on E_λ, we have
   $$A^n = \lambda^n \sum_{\ell=0}^{k-1} \binom{n}{\ell} \lambda^{-\ell} \Delta^\ell$$

   - Show that there exists a polynomial p such that
   $$\frac{||A^n||}{\lambda^n} \leq p(n) \quad \forall n \in \mathbb{N}$$

   - Show that p above can be chosen so that
   $$p(n) \leq C \frac{(|\lambda| + \frac{\delta}{2})^n}{|\lambda|^n} \quad \forall n \in \mathbb{N}$$
   for some constant $C > 0$.

- Use the previous point to prove the statement of the proposition. Hint: see lecture notes.