MA231 Honors Differential Equations — Homework 9

Sec. 4.4, variation of parameters, Green's functions, the matrix exponential

Five problems, 2 points each (10 total). Show your work. Plots may be made with Mathematica or similar.

1. (Sec. 4.4 — resonance, power and bandwidth) Consider $x'' + x' + 25x = \cos \omega t$.

(a) Find the gain $A(\omega)$ and the phase lag $\varphi(\omega)$. At what frequency ω_r is the gain largest, and what is the largest gain? What are the gain and the lag at the natural frequency $\omega = 5$?

(b) The average power the force delivers to the oscillator is $\frac{1}{2} b\, \omega^2 A(\omega)^2$. Show that for this equation it is largest at exactly $\omega = 5$, and find the largest value.

(c) Find the two half-power frequencies (where the power is half of its largest value). Check that their difference is b and that their product is k.

(d) A tuner modeled by $x'' + bx' + 25x = \cos \omega t$ must have bandwidth 0.1. What are b and the quality factor $Q = \sqrt{k}/b$?

2. (Sec. 4.4 — one complex number, and the phase plane) Consider $x'' + 2x' + 10x = 4 \cos 3t - 2 \sin 3t$.

(a) Write the forcing as $R \cos(3t - \delta)$. Give R and δ.

(b) Find the frequency response $G(3i) = 1/P(3i)$ and use it to write the steady-state solution in the form $B \cos(3t - \theta)$. Then write it as $a \cos 3t + c \sin 3t$.

(c) In the (x, x') plane, the steady state traces an ellipse. Give its semi-axes and its direction of travel, and say what every other orbit does.

(d) Now add a second forcing term: $x'' + 2x' + 10x = 4 \cos 3t - 2 \sin 3t + 2 \cos t$. Find the new steady state. Is its orbit in the (x, x') plane still an ellipse? What is its period?

3. (Variation of parameters)

(a) Find a particular solution of $y'' + 2y' + y = e^{-t} \ln t$ for $t > 0$. Check it by substituting into the equation.

(b) The functions $y_1 = t$ and $y_2 = t^2$ solve $t^2 y'' - 2ty' + 2y = 0$ for $t > 0$. Use variation of parameters to find a particular solution of $t^2 y'' - 2ty' + 2y = t^3$, and give the general solution. (*Careful:* put the equation in standard form first.)

4. (Green's functions)

(a) Let $g(\tau)$ be the solution of $y'' + 2y' + 5y = 0$ with $g(0) = 0$ and $g'(0) = 1$. Find g. Explain, in a sentence, why the solution of $y'' + 2y' + 5y = f(t)$ with $y(0) = y'(0) = 0$ is $y(t) = \int_0^t g(t-s)f(s)\,ds$.

(b) Use the formula in (a) to solve the problem with $f(t) = 1$. Check that your answer satisfies the equation and the initial conditions.

(c) Consider $y'' + 4y = f(t)$, $y(0) = y'(0) = 0$, where $f(t) = 1$ for $0 \le t \le \pi$ and $f(t) = 0$ for $t > \pi$. Use the Green's function $\frac{1}{2}\sin 2(t-s)$ to find $y(t)$ for $t \le \pi$ and for $t > \pi$. What does the answer say about the oscillator after the pulse ends, and why does this happen when the pulse lasts exactly π?

(d) *Boundary conditions.* Find the Green's function $G(t,s)$ for $-y'' = f(t)$ on $[0,1]$ with $y(0) = 0$ and $y'(1) = 0$: the function that is linear in t on each side of $t = s$, satisfies both boundary conditions, is continuous at $t = s$, and whose t-derivative drops by 1 as t passes s. Show that $G(t,s) = G(s,t)$, and use G to solve $-y'' = 1$ with these boundary conditions.

5. (The matrix exponential) Compute e^{At} for each matrix, and check that it equals I at $t = 0$.

(a) $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$ (use $A = PDP^{-1}$).

(b) $A = \begin{pmatrix} 1 & -4 \\ 1 & 1 \end{pmatrix}$ (write $A = I + K$ and compute K^2).

(c) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ (find the one eigenvalue λ, and show $(A - \lambda I)^2 = 0$).

(d) With A from (c), solve $\mathbf{Y}' = A\mathbf{Y} + \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$, $\mathbf{Y}(0) = \mathbf{0}$, using $\mathbf{Y}(t) = \int_0^t e^{A(t-s)}\mathbf{F}(s)\,ds$. Check your answer in the equation.