MA231 Honors Differential Equations — Homework 12

Power series, the Frobenius method, Bessel and Legendre equations

Five problems, 2 points each (10 total). Show your work. Plots and numerical checks may be made with Mathematica or similar.

1. (Power series at an ordinary point) Consider $y'' - ty' - y = 0$.

(a) Substitute $y = \sum a_n t^n$ and show that $a_{k+2} = \dfrac{a_k}{k+2}$. Find the first four nonzero terms of the solution with $y(0) = 1$, $y'(0) = 2$ (that is, through t^5).

(b) Identify the solution with $a_0 = 1$, $a_1 = 0$ as a function you know, and check it in the equation.

(c) Show that $y = e^{t^2/2} \displaystyle\int_0^t e^{-s^2/2}\, ds$ solves the equation, and that its series is $t + \frac{t^3}{3} + \frac{t^5}{15} + \cdots$. What is the radius of convergence of the series solutions, and why?

(d) Without solving, give a lower bound for the radius of convergence of power series solutions of $(t^2 - 2t + 5)y'' + ty' + y = 0$ about $t_0 = 0$, about $t_0 = 1$ and about $t_0 = 3$.

2. (Hermite's equation: a series that stops) Consider $y'' - 2ty' + 2ny = 0$.

(a) Show that $a_{k+2} = \dfrac{2(k-n)}{(k+1)(k+2)}\, a_k$. For $n = 5$, find the polynomial solution with $a_1 = 1$.

(b) For $n = 5$, find a_2, a_4, a_6 of the solution with $a_0 = 1$, $a_1 = 0$. Explain why this series does not stop, and why $a_{k+2}/a_k \approx 2/k$ for large k. (The coefficients of $e^{t^2} = \sum t^{2m}/m!$ have the same ratio; so this solution grows like e^{t^2}.)

(c) Let y solve Hermite's equation and set $u = e^{-t^2/2} y$. Show that $u'' = \big(t^2 - (2n+1)\big)u$. For $n = 5$ what equation does the polynomial in (a) give?

3. (Regular singular points)

(a) For each equation, put it in the form $y'' + Py' + Qy = 0$, compute $p_0 = \lim_{t\to 0} tP(t)$ and $q_0 = \lim_{t\to 0} t^2 Q(t)$, and find the indicial equation, its roots,

and which of the three cases from class applies:

$$\text{(i) } 3ty'' + 2y' + y = 0, \qquad \text{(ii) } ty'' + y' + y = 0, \qquad \text{(iii) } ty'' + 3y' + y = 0.$$

(b) Solve $4ty'' + 2y' + y = 0$ by the Frobenius method with both indicial roots, and identify the two solutions as familiar functions. Check one of them in the equation.

(c) In case (ii), the equation $ty'' + y' + y = 0$ has the solution $y_1 = \sum_{n\geq 0} \dfrac{(-t)^n}{(n!)^2}$. Find b_1, b_2, b_3 in a second solution $y_2 = y_1 \ln t + \sum_{n\geq 1} b_n t^n$.

(d) In case (iii), find the solution y_1 for the larger root (give the general coefficient). Then show that no solution of the form $t^{-2}\sum a_n t^n$ with $a_0 \neq 0$ exists, and say what this tells you about the second solution.

4. (Bessel's equation) Consider $t^2y'' + ty' + (t^2 - \nu^2)y = 0$.

(a) Show that $y = u/\sqrt{t}$ turns the equation into $u'' + \left(1 + \dfrac{\frac{1}{4} - \nu^2}{t^2}\right)u = 0$.

(b) For $\nu = \frac{1}{2}$, solve this equation for u and write the general solution y. Which solution corresponds to the indicial root $r = \frac{1}{2}$ (that is, $J_{1/2}$ up to a constant) and which to $r = -\frac{1}{2}$? Find the zeros of $J_{1/2}$.

(c) For $\nu = \frac{3}{2}$, check that $u = \dfrac{\sin t}{t} - \cos t$ solves the equation in (a). Expand u for small t and explain why $y = u/\sqrt{t}$ is the solution that goes with the indicial root $r = \frac{3}{2}$.

(d) (*The drum.*) Show that if f solves Bessel's equation with $\nu = 0$, then $y(t) = f(\omega t)$ solves $ty'' + y' + \omega^2 ty = 0$. A circular drum of radius 1 has radial vibrations $u(t) = J_0(\omega t)$ with $u(1) = 0$ (the edge is held fixed). Given that the first two zeros of J_0 are 2.4048 and 5.5201, find the two lowest radial frequencies ω and their ratio.

5. (Legendre's equation) Consider $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$.

(a) For $n = 4$, use the recurrence $a_{k+2} = \dfrac{(k-4)(k+5)}{(k+1)(k+2)}\, a_k$ to find the even polynomial solution, and normalize it so that its value at $x = 1$ is 1. Check that you get $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.

(b) Compute directly $\displaystyle\int_{-1}^{1} P_4(x)P_2(x)\,dx$ and $\displaystyle\int_{-1}^{1} P_4(x)^2\,dx$, where $P_2 = \frac{1}{2}(3x^2 - 1)$.

(c) Expand x^4 in Legendre polynomials, using $c_n = \dfrac{2n+1}{2}\displaystyle\int_{-1}^{1} x^4 P_n(x)\,dx$. Check your answer by multiplying out.

(d) For $n = 0$ the equation is $\left((1 - x^2)y'\right)' = 0$. Find both solutions, and show that one of them is unbounded as $x \to 1^-$ and as $x \to -1^+$. How does this explain why only integer n give a solution bounded at both ends?